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AXI-SYMMETRIC DYNAMIC RESPONSE OF CIRCULAR PLATES WITH ARBITRARY ELASTIC BOUNDARY RESTRAINTS†

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Abstract—The dynamic response of circular plates to axi-symmetric time-dependent loads is studied. The plate is analyzed according to classical plate theory, neglecting rotatory inertia and shear strains. The solution is obtained for a plate with axi-symmetric but arbitrary linear elastic restraints at the outer boundary. The edge restraints are with respect to both rotational and transverse motions. Particular attention is directed to a study of the effect of the transverse restraint on the maxima of the response. As a practical example, numerical results are presented for a plate subjected to an N-shaped pressure wave resulting from a sonic boom.

NOTATION

a	radius of plate
h	thickness of plate
$q_n(t)$	generalized coordinate
r	radial distance
t	time
w	displacement
D	$= \dfrac{Eh^3}{12(1-\nu^2)}$, flexure rigidity of plate
E	modulus of elasticity
J_n, I_n	Bessel function and Modified Bessel function of first kind, respectively
M_r, M_t, Q_r	Moments (radial and tangential) and shear force
Q_n	generalized force
$\tilde{q}_n, \tilde{w}, \tilde{M}_r, \tilde{M}_t, \tilde{Q}_r$	non-dimensionalized quantities
α, β	restraining spring constants
ψ_n	modal shapes
ζ_n	eigenvalue
$\rho = r/a$	non-dimensional radial distance
$\bar{\rho}$	mass density of plate
μ_n	normalizing factor, generalized mass factor
η	non-dimensional time
ν	Poisson's ratio
ω	circular frequency
$(\ )', (\dot{\ })$	differentiation with respect to ρ and t respectively

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1. INTRODUCTION

FLEXURAL vibrations of circular elastic plates have been studied by various investigators starting with Poisson and Kirchhoff. In more recent studies, Reismann [1] treated the case of harmonically oscillating concentrated loads applied to a fully clamped circular plate, while Kantham [2] obtained natural frequencies and mode shapes for elastically built-in plates. The forced axi-symmetric vibrations of an elastic circular plate with elastic rotational restraints at the edge but with rigid transverse supports were considered by Weiner [3]. The dynamic response of various structural elements, particularly plates subjected to sonic boom shock waves, has also been the subject of several investigations [4, 5].

The problem considered here is the investigation of forced vibrations of elastic circular plates with arbitrary (linear) elastic restraints at the edge against both rotational and transverse displacements. The outer edge is assumed to be resting on spring supports rather than on rigid supports, thus permitting transverse as well as rotational motion to occur. In addition, the edge is restrained by torsional springs which restrain the edge rotation. By varying the relevant parameters, extreme cases of rigid supports are obtained, thus recovering the solution given by Weiner in [3].

The dynamic response to axi-symmetric transverse loads with arbitrary time dependency is obtained by means of a modal analysis using small strain theory and neglecting rotatory inertia and shear strains.

Since it is desired to obtain the effect of relaxing the restraint against transverse motion at the edge, emphasis on this aspect of the problem is given in the results.

Numerical results are presented, in particular, for the dynamic response of circular plates subjected to sonic booms. The maxima of the response are given and, in order to provide a comparison with the corresponding maximum static quantities, the results are also given in terms of dynamic amplification factors.

2. FORCED VIBRATIONS OF ELASTICALLY RESTRAINED CIRCULAR PLATES

The problem considered is the response of an elastic circular plate subjected to time-dependent axi-symmetric loads $P(r, t)$. The plate is restrained at the outer boundary $r = a$ by means of supports which resist elastically both transverse displacements and rotations, and may be represented by the model shown in Fig. 1 where α and β are linear spring constants.

The classical equation of motion governing flexural vibrations (neglecting shear deformations and rotatory inertia) is

$$D\nabla^4 w(r, t) + \bar{\rho} h \ddot{w}(r, t) = P(r, t). \tag{2.1a}$$

Introducing a non-dimensional radial distance $\rho \equiv r/a$, the equation of motion may be written as

$$\nabla^4 w(\rho, t) + \frac{\bar{\rho} h a^4}{D} \ddot{w}(\rho, t) = \frac{a^4}{D} P(\rho, t) \tag{2.1b}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho}. \tag{2.2}$$

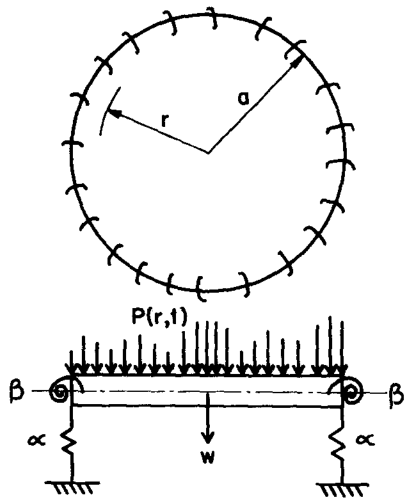

FIG. 1. Geometry of the problem.

The associated boundary conditions at the outer edge ($\rho = 1$) are

$$Q_r(1) = -\alpha w(1), \qquad M_r(1) = \frac{\beta}{a} w'(1). \tag{2.3}$$

Using the well-known expressions, given e.g. in [6], these boundary conditions become

$$\begin{aligned} \frac{D}{\alpha a^3}\left(w''' + \frac{1}{\rho} w'' - \frac{1}{\rho^2} w' \right)\bigg|_{\rho=1} &= w(1) \\ \frac{-D}{\beta a}\left(w'' + \frac{\nu}{\rho} w' \right)\bigg|_{\rho=1} &= w'(1) \end{aligned} \tag{2.4†}$$

where primes indicate differentiation with respect to ρ.

Furthermore, at the origin $\rho = 0$, conditions on the bounded displacements and on axial symmetry give

$$|w(0, t)| \leq M$$

and (2.5)

$$w'(0, t) = 0.$$

The initial conditions are prescribed as

$$w(\rho, 0) = \frac{\partial w}{\partial t}(\rho, 0) = 0. \tag{2.6}$$

The problem will be solved by means of a modal analysis. To this end, the free vibrations are first considered.

† Note that the sign convention adopted for Q_r is positive downward on a positive r-surface and differs from that given in [6].

Assuming free vibrations of the form

$$w(\rho, t) = \psi(\rho)\, e^{i\omega t}, \tag{2.7}$$

the resulting homogeneous equation of motion is

$$(\nabla^4 - \zeta^4)\psi(\rho) = 0, \tag{2.8}$$

where

$$\zeta^4 \equiv \bar{\rho} h a^4 \omega^2 / D. \tag{2.9}$$

From (2.4) and (2.5) the conditions on $\psi(\rho)$ become

$$\begin{aligned} \frac{D}{\alpha a^3}\left(\psi''' + \frac{1}{\rho}\psi'' - \frac{1}{\rho^2}\psi'\right)\Bigg|_{\rho=1} &= \psi(1) \\ \frac{-D}{\beta a}\left(\psi'' + \frac{\nu}{\rho}\psi'\right)\Bigg|_{\rho=1} &= \psi'(1) \\ |\psi(0)| &\leq M \\ \psi'(0) &= 0. \end{aligned} \tag{2.10}$$

The general solution of (2.8) is

$$\psi(\rho) = AJ_0(\zeta\rho) + BI_0(\zeta\rho) + CY_0(\zeta\rho) + DK_0(\zeta\rho). \tag{2.11a}$$

However, making use of the last two of (2.10), $C = D = 0$, since the Bessel functions of the second kind go independently to infinity at the origin; therefore

$$\psi(\rho) = AJ_0(\zeta\rho) + BI_0(\zeta\rho). \tag{2.11b}$$

Substituting (2.11b) in the remaining boundary conditions of (2.10) and using the recurrence relations for Bessel functions [7], the relations between the constants A and B are established:

$$\begin{aligned} [J_0(\zeta) - \Gamma s\zeta^3 J_1(\zeta)]A + [I_0(\zeta) - \Gamma s\zeta^3 I_1(\zeta)]B &= 0 \\ [\zeta J_0(\zeta) + (\Gamma_R - 1)J_1(\zeta)]A + [-\zeta I_0(\zeta) - (\Gamma_R - 1)I_1(\zeta)]B &= 0 \end{aligned} \tag{2.12}$$

where

$$\Gamma_s \equiv \frac{D}{\alpha a^3}, \qquad \Gamma_R \equiv \frac{\beta a}{D} + \nu \tag{2.13}$$

are non-dimensional constants.

For a non-trivial solution to exist, the determinant of the coefficients must vanish, from which the frequency equation,

$$2\zeta J_0(\zeta)I_0(\zeta) - 2\Gamma_s(\Gamma_R - 1)\zeta^3 J_1(\zeta)I_1(\zeta) + (\Gamma_R - 1 - \Gamma_s\zeta^4)[J_0(\zeta)I_1(\zeta) + J_1(\zeta)I_0(\zeta)] = 0 \tag{2.14}$$

is obtained.

The roots (eigenvalues)

$$\zeta_n, \qquad n = 1, 2, 3, \ldots$$

are given in Table 1 and the variation with the specified boundary conditions is shown in Fig. 2. The corresponding frequencies ω_n, from (2.9) are,

$$\omega_n = \frac{\zeta_n^2}{a^2}\sqrt{\left(\frac{D}{\bar{\rho}h}\right)} \tag{2.15}$$

while the corresponding eigenfunctions representing the mode shapes become†

$$\psi_n(\rho) = [I_0 - \Gamma_s\zeta_n^3 I_1]J_0(\zeta_n\rho) - [J_0 - \Gamma_s\zeta_n^3 J_1]I_0(\zeta_n\rho) \tag{2.16}$$

TABLE 1. TABLE OF EIGENVALUES, ζ_n

		Γ_R				
n	Γ_s	0·25	1·0	10	100	∞
1	0	2·2046	2·4048	2·9529	3·1652	3·19622
2		5·4463	5·5201	5·9280	6·2471	6·30643
3		8·6082	8·6537	8·9762	9·3534	9·4395
4		11·7586	11·7915	12·0570	12·4657	12·5771
5		14·9051	14·9309	15·1560	15·5813	15·7164
6		18·0499	18·0711	18·2662	18·6990	18·8565
7		21·1936	21·2116	21·3838	21·8184	21·9971
8		24·3368	24·3525	24·5064	24·9393	25·1379
9		27·4796	27·4936	27·6325	28·0616	28·2787
10		30·6167	30·6243	30·7576	31·1808	31·4158
1	0·50	1·3728	1·3853	1·4028	1·4064	1·4068
2		3·0659	3·2575	3·6982	3·8317	3·8496
3		6·2003	6·3144	6·7642	6·9848	7·0184
4		9·3644	9·4419	9·8466	10·1262	10·1744
5		12·5196	12·5781	12·9398	13·2617	13·3241
6		15·6699	15·7170	16·0419	16·3949	16·4709
7		18·8176	18·8568	19·1510	19·5269	19·6160
8		21·9636	21·9973	22·2656	22·6583	22·7602
9		25·1085	25·1380	25·3844	25·7894	25·9036
10		28·2525	28·2785	28·5076	28·9206	29·0467
1	1·0	1·1715	1·1769	1·1844	1·1859	1·1861
2		3·0239	3·2269	3·6851	3·8223	3·8406
3		6·1959	6·3104	6·7619	6·9833	7·0170
4		9·3631	9·4407	9·8458	10·1256	10·1739
5		12·5191	12·5776	12·9394	13·2615	13·3239
6		15·6697	15·7167	16·0417	16·3948	16·4707
7		18·8174	18·8567	19·1509	19·5268	19·6159
8		21·9635	21·9972	22·2655	22·6583	22·7601
9		25·1084	25·1380	25·3843	25·7895	25·9037
10		28·2524	28·2786	28·5065	28·9207	29·0474
1	2·0	0·9925	0·99481	0·99798	0·9986	0·9987
2		3·0028	3·21154	3·6786	3·8176	3·8362
3		6·1937	6·3084	6·7607	6·9825	7·0163
4		9·3625	9·4401	9·8453	10·1254	10·1737
5		12·5188	12·5774	12·9392	13·2614	13·3238
6		15·6695	15·7166	16·0416	16·3947	16·4707
7		18·8173	18·8566	19·1508	19·5268	19·6159
8		21·9634	21·9971	22·2655	22·6583	22·7601
9		25·1084	25·1379	25·3843	25·7894	25·9037
10		28·2524	28·2790	28·5066	28·9197	29·0457

† For ease of notation, Bessel functions J and I appearing in equation (2.16) and all subsequent equations, without any indicated arguments denote argument at $\rho = 1$, i.e. $J_0 \equiv J_0(\zeta_n)$, $I_0 \equiv I_0(\zeta_n)$, etc.

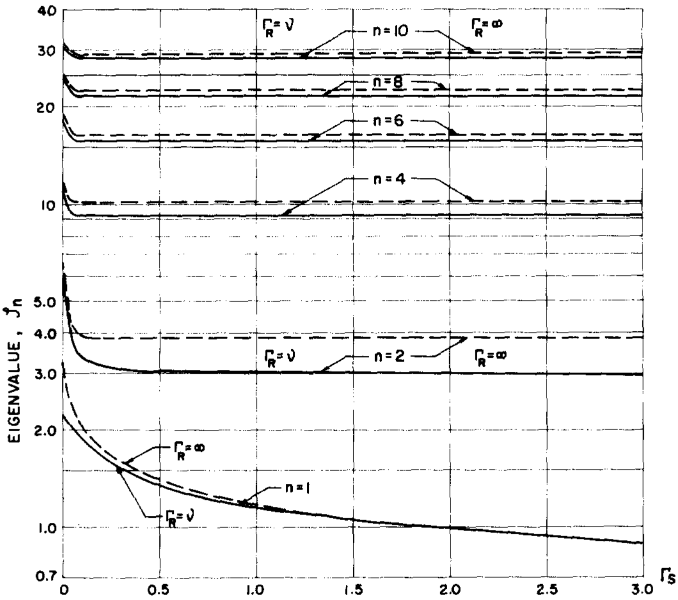

FIG. 2. Eigenvalues ζ_n for $\Gamma_R = \nu$, $\Gamma_R = \infty$; $\nu = 0{\cdot}25$.

The forced displacements $w(\rho, t)$, produced by axi-symmetric forces, are now assumed to be represented by the series

$$w(\rho, t) = \sum_{n=1}^{\infty} \psi_n(\rho) q_n(t) \tag{2.17}$$

where $q_n(t)$ are generalized coordinates.

Substituting this expression in (2.1b)

$$\sum_{n=1}^{\infty} \left[\nabla^4 \psi_n q_n + \frac{\bar{\rho} h a^4}{D} \psi_n \ddot{q}_n \right] = \frac{a^4}{D} P(\rho, t). \tag{2.18}$$

Multiplying through by ψ_m, summing over m, integrating over the domain $0 \leq \rho \leq 1$, yields (after interchanging the summation and integration processes):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \int_0^1 \left[\psi_m \nabla^4 \psi_n q_n + \frac{\bar{\rho} h a^4}{D} \psi_m \psi_n \ddot{q}_n \right] \rho \, \mathrm{d}\rho \right\} = \sum_{m=1}^{\infty} \frac{a^4}{D} \int_0^1 \psi_m(\rho) P(\rho, t) \rho \, \mathrm{d}\rho. \tag{2.19}$$

From (2.8) and (2.15) there results

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ [\ddot{q}_n(t) + \omega_n^2 q_n(t)] \int_0^1 \psi_m \psi_n \rho \, \mathrm{d}\rho \right\} = \sum_{m=1}^{\infty} \frac{1}{\bar{\rho} h} \int_0^1 \psi_m(\rho) P(\rho, t) \rho \, \mathrm{d}\rho \tag{2.20}$$

By virtue of the orthogonality relation (see Appendix A),

$$\int_0^1 \psi_m(\rho) \psi_n(\rho) \rho \, \mathrm{d}\rho = \begin{cases} \mu_n, & n = m \\ 0, & n \neq m \end{cases} \tag{2.21}$$

the uncoupled equations on the generalized coordinates become

$$\ddot{q}_n(t)+\omega_n^2 q_n = \frac{Q_n}{\bar{\rho}h\mu_n} \tag{2.22}$$

where

$$Q_n(t) = \int_0^1 \psi_n(\rho)P(\rho,t)\rho\,\mathrm{d}\rho. \tag{2.23}$$

It may be noted that $\bar{\rho}h\mu_n$ and Q_n are effectively the generalized mass and force which could equally well have been derived using the Lagrange equations of motion governing the system.

The general solution of (2.22) for the initial conditions (2.6) is then

$$q_n(t) = \frac{1}{\bar{\rho}h\mu_n\omega_n}\int_0^t Q_n(t)\sin\omega_n(t-\tau)\,\mathrm{d}\tau \tag{2.24}$$

and the displacements are given by (2.17).

Expressions for the moment and shear resultant force systems can then be obtained from the expressions [6]

$$\begin{aligned} M_r(\rho,t) &= -\frac{D}{a^2}\left(w''+\frac{v}{\rho}w'\right)\\ M_t(\rho,t) &= -\frac{D}{a^2}\left(vw''+\frac{1}{\rho}w'\right)\\ Q_r(\rho,t) &= \frac{D}{a^3}\left(w'''+\frac{1}{\rho}w''-\frac{1}{\rho^2}w'\right). \end{aligned} \tag{2.25}$$

Substituting (2.17) and (2.16) into the preceding equations, after some algebraic manipulation, results in the following

$$\begin{aligned} M_r(\rho,t) &= \frac{D}{a^2}\sum_{n=1}^{\infty} h_n^{(r)}(\rho)q_n(t)\\ M_t(\rho,t) &= \frac{D}{a^2}\sum_{n=1}^{\infty} h_n^{(t)}(\rho)q_n(t)\\ Q_r(\rho,t) &= \frac{D}{a^3}\sum_{n=1}^{\infty} h_n^{(Q)}(\rho)q_n(t). \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} h_n^{(r)}(\rho) &= \zeta_n\left\{\zeta_n[G_1I_0(\zeta_n\rho)+G_2J_0(\zeta_n\rho)]+\left(\frac{1-v}{\rho}\right)[G_1I_1(\zeta_n\rho)+G_2J_1(\zeta_n\rho)]\right\}\\ h_n^{(t)}(\rho) &= \zeta_n\left\{v\zeta_n[G_1I_0(\zeta_n\rho)+G_2J_0(\zeta_n\rho)]+\left(\frac{1-v}{\rho}\right)[G_1I_1(\zeta_n\rho)+G_2J_1(\zeta_n\rho)]\right\}\\ h_n^{(Q)}(\rho) &= \zeta_n^3[G_1I_1(\zeta_n\rho)-G_2J_1(\zeta_n\rho)] \end{aligned} \tag{2.27}$$

and

$$G_1 = J_0 - \Gamma_s \zeta_n^3 J_1, \qquad G_2 = I_0 - \Gamma_s \zeta_n^3 I_1. \tag{2.28}$$

Terms such as $[J_1(\zeta_n\rho)/\rho]$, $[I_1(\zeta_n\rho)/\rho]$ produce no singularities at the origin since

$$\lim_{\rho \to 0} \frac{J_1, I_1(\zeta_n \rho)}{\rho} = \frac{\zeta_n \rho}{2\Gamma(2)\rho} = \frac{\zeta_n}{2}.$$

It may be noted that the solution degenerates to that given in [3] by letting $\Gamma_s = 0$ for a plate fully constrained against transverse motion at the outer edge.

The case of a simple support and a clamped edge is obtained by setting $\Gamma_R = \nu$ or $\Gamma_R = \infty$, respectively.

3. UNIFORMLY DISTRIBUTED TIME-DEPENDENT LOADS: RESPONSE TO A SONIC BOOM

For the case of applied loads $P(t)$ which are not space-dependent, the expression for the generalized force $Q_n(t)$ becomes, upon substitution of (2.16) in (2.23),

$$Q_n(t) = P(t) g_n(\zeta_n) \tag{3.1}$$

where

$$g_n(\zeta_n) = \frac{1}{\zeta_n}[(I_0 - \Gamma_s \zeta_n^3 I_1)]J_1 - (J_0 - \Gamma_s \zeta_n^3 J_1) I_1]. \tag{3.2}$$

As an example of some practical interest, the response of a circular plate to a particular pressure $P(t)$, resulting from a sonic boom, will be considered. It has been shown that the applied pressure-time history at far fields due to an aircraft flying at a supersonic velocity V (for which the effect of the boom is felt over a distance L_s) may be approximated closely by means of an N-shape pulse [8] (Fig. 3):

$$P(t) = P_0(1 - 2t/T^*), \qquad t < T^* \tag{3.3}$$

where T^*, the time duration of the pressure, is given by

$$T^* = L_s/V \tag{3.4}$$

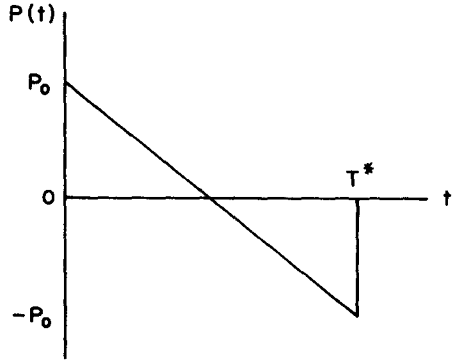

FIG. 3. Idealized N-shape pulse.

Introducing the non-dimensional variables

$$\eta \equiv t/T^*, \qquad \xi_n \equiv \omega_n T^*, \qquad \tilde{q}_n \equiv \frac{Eq_n}{P_0 h}, \tag{3.5}$$

substituting (3.1) and (3.3) in (2.24) and performing the indicated integration, results in the following expression for the generalized coordinates

$$\tilde{q}_n = \frac{12(1-\nu^2)g_n(\zeta_n)}{\zeta_n^4 \mu_n}(a/h)^4 f(\eta) \tag{3.6}$$

where

$$f(\eta) = \begin{cases} 1-\cos \xi_n\eta - \dfrac{2}{\xi_n}(\xi_n\eta - \sin \xi_n\eta) & \eta \le 1 \\ -\cos \xi_n(\eta-1) - \cos \xi_n\eta - \dfrac{2}{\xi_n}[\sin \xi_n(\eta-1) - \sin \xi_n\eta], & \eta \ge 1 \end{cases} \tag{3.7}$$

and

$$\xi_n = \frac{\zeta_n^2}{a}(h/a)T^*\left[\frac{E}{12(1-\nu^2)\bar{\rho}}\right]^{\frac{1}{2}}. \tag{3.8}$$

Noting that the expression for the velocity of longitudinal waves in a plate

$$C_L = \left[\frac{E}{(1-\nu^2)\bar{\rho}}\right]^{\frac{1}{2}} \tag{3.9a}$$

appears in (3.8), and defining the critical time required for a wave to travel the distance of one radius as

$$T_{cr} = a/C_L, \tag{3.9b}$$

it is observed that the generalized coordinates depend solely on a set of non-dimensional variables, viz.

$$\tilde{q}_n = \tilde{q}_n[h/a, \Gamma_S, \Gamma_R, \nu, T^*/T_{cr}; \eta]. \tag{3.10}$$

The resulting non-dimensional deflections are then

$$\tilde{w}(\rho, \eta) \equiv \frac{Ew}{P_0 h} = \sum_{n=1}^{\infty} \psi_n(\rho)\tilde{q}_n(\eta). \tag{3.11}$$

Similarly, the non-dimensional moments and shear forces may be expressed as

$$\begin{aligned} \tilde{M}_r, \tilde{M}_t(\rho, \eta) &\equiv \frac{M_r, M_t}{P_0 a^2} = K \sum_{n=1}^{\infty} [h_n^{(r)}, h_n^{(t)}(\rho)]\tilde{q}_n(\eta) \\ \tilde{Q}_r &\equiv \frac{Q_r}{P_0 a} = K \sum_{n=1}^{\infty} h_n^{(Q)}(\rho)\tilde{q}_n(\eta) \end{aligned} \tag{3.12}$$

where $h_n^{(r)}, h_n^{(t)}, h_n^{(Q)}$ are given by (2.27) and

$$K = \frac{(h/a)^4}{12(1-\nu^2)}. \tag{3.13}$$

4. NUMERICAL RESULTS FOR A TYPICAL CASE

Numerical results are presented for the response of a circular plate to N-shaped pressure waves of time duration T^* in the range $0{\cdot}1 \leq T^* \leq 0{\cdot}4$ seconds. The lower values represent the duration of pressure waves due to supersonic military aircraft while the upper range represents waves expected from future supersonic transports.

Results are shown for plates which are clamped at the outer edge against rotation ($\Gamma_R = \infty$) as well as for plates having no rotational restraint at the edges ($\Gamma_R = \nu$). The responses for intermediate moment restraints were found to vary within the given range. The results are also presented for circular plates with transverse restraints given in the range of the governing parameters, $0 \leq \Gamma_S \leq 3{\cdot}5$, where the lower value $\Gamma_S = 0$ represents a rigid (transverse) support.

The results are given for a plate with $\nu = 0{\cdot}25$, a geometric ratio $h/a = 8{\cdot}33 \times 10^{-3}$, and are presented for a range of the nondimensional ratio $350 \leq T^*/T_{cr} \leq 1400$.†

The response was calculated from a modal analysis in which the first ten modes were considered. However, it was found that the convergence of the series for displacements and internal force resultants is very rapid and a summation of the first 5 or 6 modes is usually sufficient to provide answers within an accuracy of 2%. Moreover, it has been observed that the response is largely dependent on the fundamental mode $n = 1$.

From Fig. 2, it is evident that the eigenvalue ζ_1 and therefore the fundamental frequency ω_1 is extremely sensitive to the transverse restraint as governed by the parameter Γ_s (particularly in the lower range of Γ_s), decreasing rapidly with an increase in Γ_s.

The largest transverse displacement of the plate was found to occur always at the center point ($\rho = 0$). In fact, the displacement profile was found to depend very weakly on the parameter Γ_s. A typical time history of the displacement for plates with no rotational restraint is given in Fig. 4 for a range of values of Γ_s. Plates with complete rotational fixity (clamped) at the edges ($\Gamma_R = \infty$) show a similar time-history.

The maximum displacement of the plate within the given range of T^*/T_{cr}, for both free and clamped plates, is given in Fig. 5 as a function of Γ_s. For comparison, the equivalent static displacements are shown superimposed on the Figure. It is to be noted that, for the lower range of values of Γ_s, the dynamic response is always greater than the static response. However, for sufficiently large values of Γ_s, i.e. with sufficiently soft translational springs, the dynamic response can always be attenuated causing the dynamic displacements to be considerably smaller than the equivalent static displacements.

As seen from Fig. 5, the responses for the plate with no rotational restraint ($\Gamma_R = \nu$) and the clamped plate ($\Gamma_R = \infty$) follow the same pattern. Further results, therefore, are presented only for the first case. (More extensive results may be found in [9]).

The preceding results are demonstrated concisely for all given values of T^*/T_{cr} in Fig. 6 by means of the Dynamic Amplification Factors (DAF) for the displacements, which are defined as the ratios of maximum dynamic to maximum equivalent static response.

Similar calculations are presented for the resultant internal force systems; Figs. 7 and 8 show the results obtained for the bending moments and shear forces respectively. Corresponding dynamic amplification factors are also given.

† These values correspond to a typical glass pane of radius $a = 60$ in., $h = 0{\cdot}5$ in., with material properties $E = 10 \times 10^6$ lb/in^2, $\bar{\rho} = 2{\cdot}4 \times 10^{-4}$ lb sec^2/in^4. These values are representative of usual glass panes in actual use at present times. The range of T^*/T_{cr} corresponds to the range of pressure time duration due to sonic booms as given above.

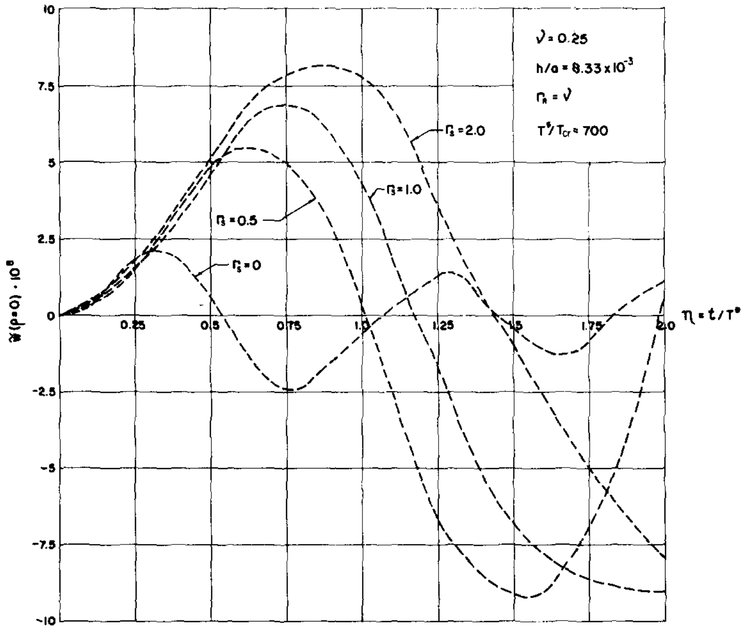

FIG. 4. Typical center displacement time variation for N-pulse input.

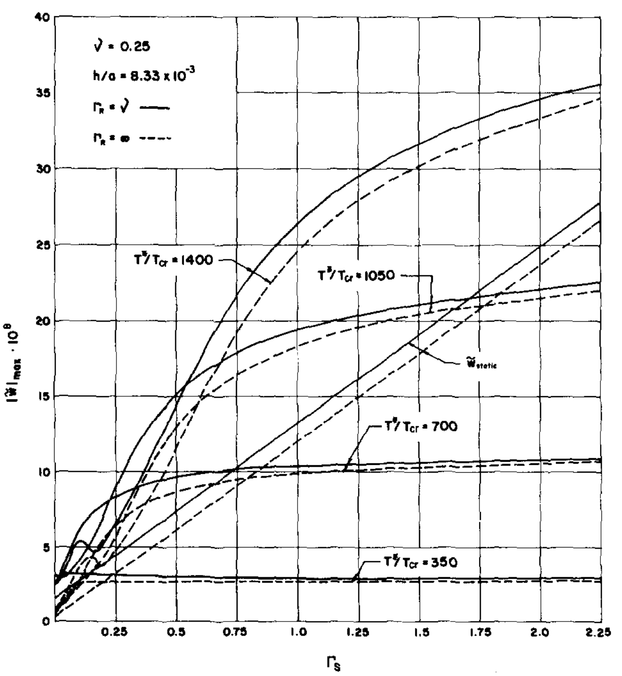

FIG. 5. Variation of maximum displacement with transverse restraint.

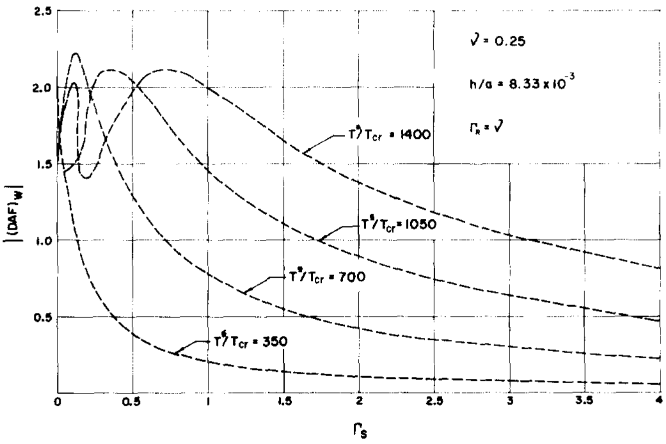

FIG. 6. Dynamic amplication factor for displacements of plate with no rotational restraint.

As mentioned above, it was found that the response in all cases was reflected primarily in the response of the fundamental mode. By plotting the maximum transverse displacement (and corresponding DAF) as a function of $(T^*/T_{cr})/(\tau_1/T_{cr})$, (where τ_1 is the fundamental period of the plate), it is seen that the response is highly dependent on this parameter as shown in Fig. 9 for the case of free plates. One observes, e.g. that the DAF for the displacement becomes smaller than unity whenever $T^*/\tau_1 \leq 0{\cdot}4$. Similar results are obtained for moments as shown in Fig. 10.

5. CONCLUSIONS

The forced axi-symmetric vibration solution of an elastic plate with arbitrary elastic boundary restraints against both transverse and rotational motion has been obtained. As a practical illustration, the dynamic response of a plate subjected to a sonic boom was determined.

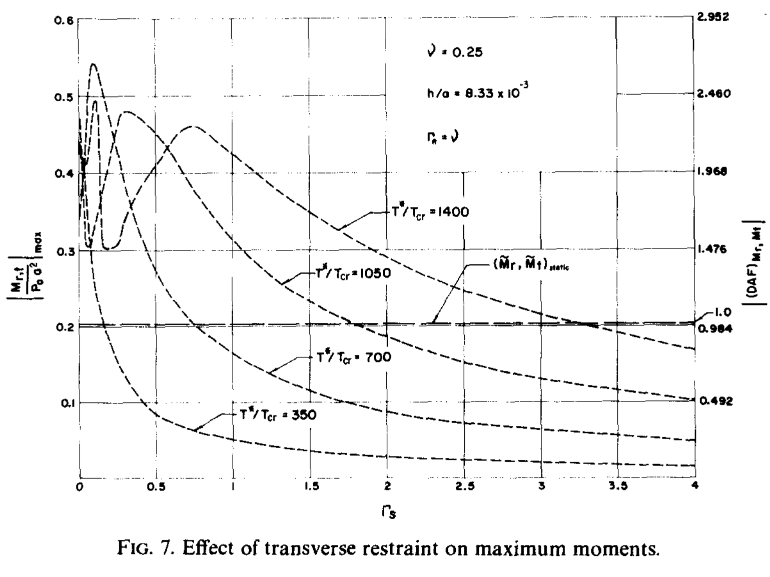

FIG. 7. Effect of transverse restraint on maximum moments.

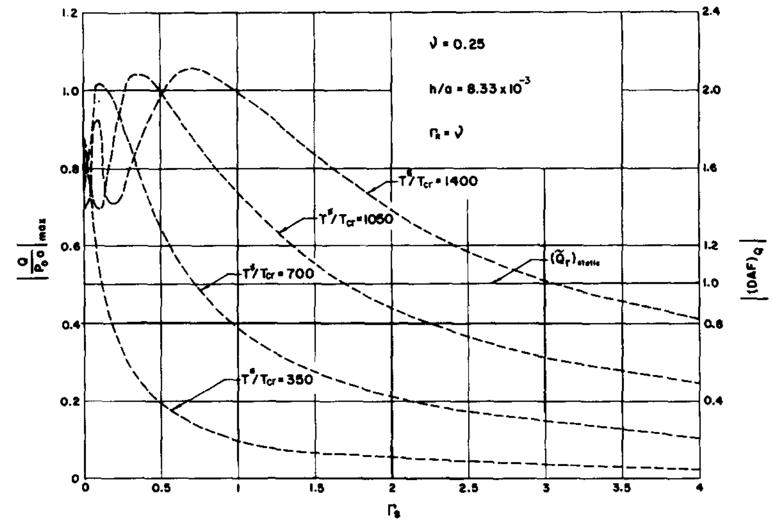

FIG. 8. Effect of transverse restraint on maximum shear.

The response is seen to depend predominantly on the fundamental mode. While the displacement profile was found to depend very weakly on the transverse restraint, the critical values of maximum response are observed to be strongly dependent on the relative transverse restraint while being weakly dependent on the rotational restraint at the boundary.

From a study of the variation of the response with a relaxation of the transverse edge support, it may be concluded that a moderate flexibility of this restraint can increase the fundamental period sufficiently and thereby considerably attenuate the maximum dynamic response.

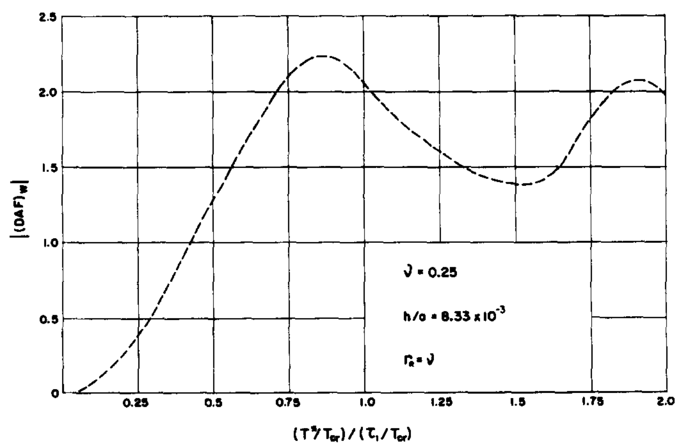

FIG. 9. Variation of maximum displacement DAF with ratio T^*/τ_1 ; $\Gamma_R = \nu$.

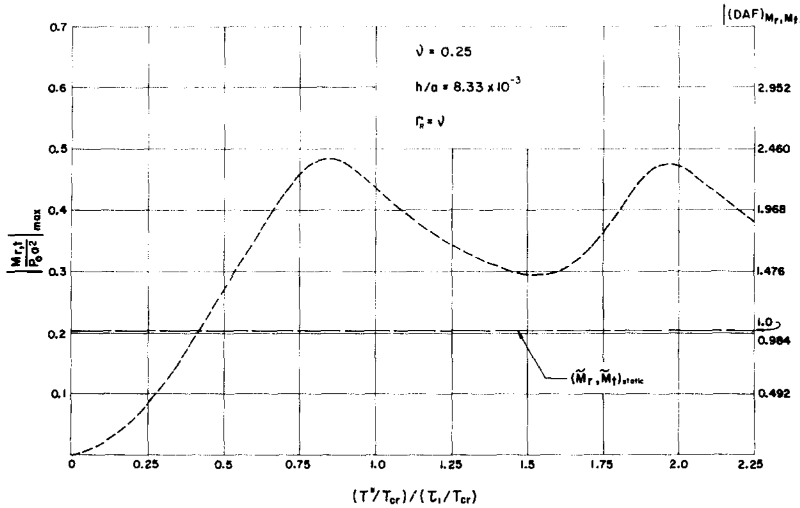

FIG. 10. Variation of maximum moments with ratio T^*/τ_1 ; $\Gamma_R = \nu$.

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APPENDIX A

Orthogonality and normalizing factor for the eigenfunctions, ψ_n

From (2.8), the functions $\psi_n(\rho)$ satisfy the equations

$$\nabla^4\psi_n = \zeta_n^4\psi_n \quad \text{or} \quad \nabla^4\psi_m = \zeta_m^4\psi_m. \tag{A.1}$$

Multiplying the first of the above equations by ψ_m and the second by ψ_n and subtracting the two resulting equations, one from the other, there results

$$\psi_m\nabla^4\psi_n - \psi_n\nabla^4\psi_m = (\zeta_n^4 - \zeta_m^4)\psi_n\psi_m. \tag{A.2}$$

Consider the integral over the domain:

$$\mu_{mn} = \int_0^1 \psi_m\nabla^4\psi_n\rho\,\mathrm{d}\rho.$$

Since the bi-harmonic operator ∇^4 may be written, for the axi-symmetric case as

$$\nabla^4 \equiv \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left\{\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\right)\right]\right\}$$

it follows that

$$\mu_{mn} = \int_0^1 \psi_m(\rho)\,\mathrm{d}\left\{\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}\psi_n}{\mathrm{d}\rho}\right)\right]\right\} \tag{A.3}$$

Integrating twice by parts

$$\mu_{mn} = \bar{A}_{mn} - \bar{B}_{mn} + \int_0^1 \frac{1}{\rho}[\psi'_n + \rho\psi''_n][\psi'_m + \rho\psi''_m]\,\mathrm{d}\rho \tag{A.4}$$

where

$$\begin{aligned} \bar{A}_{mn} &\equiv \psi_m\left(-\frac{\psi'_n}{\rho} + \psi''_n + \rho\psi'''_n\right)\Bigg|_0^1 \equiv \bar{A}_{mn}(1) - \bar{A}_{mn}(0) \\ \bar{B}_{mn} &\equiv \psi'_m(\psi'_n + \rho\psi''_n)\Big|_0^1 \equiv \bar{B}_{mn}(1) - \bar{B}_{mn}(0). \end{aligned} \tag{A.5}$$

Now

$$\bar{A}_{mn}(0) = \psi_m(\rho)\left(-\frac{\psi'_n}{\rho} + \psi''_n\right)\Bigg|_{\rho=0} = \rho\psi_m\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\psi'_n}{\rho}\right) = 0$$

since $\lim_{\rho\to 0}\psi'_m(\rho) = \lim_{\rho\to 0}(\zeta_n\rho) = 0$. Similarly $\bar{B}_{mn}(0) = 0$.

To evaluate $\bar{A}_{mn}(1)$ and $\bar{B}_{mn}(1)$, use is made of the boundary conditions, (2.10). Substituting the equations in (A.5), and from the definition of (2.13),

$$\begin{aligned} \bar{A}_{mn}(1) &= \Gamma_s\psi_m(1)\psi_n(1) \\ \bar{B}_{mn}(1) &= (1-\Gamma_R)\psi'_n(1)\psi'_m(1). \end{aligned} \tag{A.6}$$

Thus one notices that $\bar{A}_{mn}$, $\bar{B}_{mn}$, and the integral appearing in (A.4) are all symmetric in m and n. Hence it follows that $\mu_{mn} = \mu_{nm}$, and therefore the orthogonality condition

$$\int_0^1 \psi_m(\rho)\psi_n(\rho)\rho\,\mathrm{d}\rho = 0, \qquad m \neq n \tag{A.7}$$

is established.

The normalizing factor

$$\mu_n = \int_0^1 \psi_n^2(\rho)\,\mathrm{d}\rho \tag{A.8}$$

is then, from (A.4) and (A.6),

$$\mu_n = \Gamma_s\psi_n^2(1) - (1-\Gamma_R)[\psi'_n(1)]^2 + \int_0^1 \frac{1}{\rho}[\psi'_n(\rho) + \rho\psi''_n(\rho)]^2\,\mathrm{d}\rho. \tag{A.9}$$

Substituting (2.16) and making use of the relations provided by the frequency equation (2.14), after some algebraic manipulation, the normalizing factor becomes

$$\mu_n = \tfrac{1}{2}[(J_0^2+J_1^2)(I_0-\Gamma_s\zeta_n^3 I_1)^2+(J_0-\Gamma_s\zeta_n^3 J_1)^2(I_0^2-I_1^2)] \\ +\frac{\Gamma_s\zeta_n^2}{2}(J_1 I_0-J_0 I_1)^2+\frac{2(\Gamma_R-1)}{(\Gamma_R-1-\Gamma_s\zeta_n^4)^2}\cdot(J_0 I_0-\Gamma_s^2\zeta_n^6 J_1 I_1)^2. \tag{A.10}$$

APPENDIX B

Static solution

In order to determine the dynamic amplification factors for the sonic boom pressure, (as defined in Section 4)

$$\text{DAF} = \frac{(\text{Dynamic Response})_{\max}}{(\text{Static Response})_{\max}}$$

the static solution to the corresponding problem is needed.

For convenience, the static solution of the elastically restrained plate subjected to a uniformly distributed load P_0 is given in terms of the parameters used in this paper. The solution, modified for rigid body translation, is based on the results given in [2].

The relevant quantities are given as follows:

$$\tilde{W}_{\text{static}} = \left(\frac{Ew}{P_0 h}\right)_{\text{static}} = 6(1-\nu^2)(a/h)^4\{\tfrac{1}{32}[\rho^4-\rho^2(N+1)+N]+\Gamma_s\} \tag{B.1}$$

$$\tilde{M}_{r_{\text{static}}} = \left(\frac{M_r}{P_0 a^2}\right)_{\text{static}} = -\tfrac{1}{32}[2\rho^2(\nu+3)-(1+\nu)(N+1)] \tag{B.2}$$

$$\tilde{M}_{t_{\text{static}}} = \left(\frac{M_t}{P_0 a^2}\right)_{\text{static}} = -\tfrac{1}{32}[2(1+3\nu)\rho^2-(1+\nu)(N+1)] \tag{B.3}$$

$$\tilde{Q}_{r_{\text{static}}} = \left(\frac{Q_r}{P_0 a}\right)_{\text{static}} = -\rho/2 \tag{B.4}$$

where

$$N = \frac{5+\Gamma_R}{1+\Gamma_R} \tag{B.5}$$

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Абстракт—Исследуется динамическое поведение круглых пластинок под влиянием осесимметричес-кой, зависящей от времени нагрузки. Приводится анализ пластинки согласно классической теории пластинки, пренебрегая инерцию вращения и деформации сдвига. Получается решение для пластинки с осесимметрическими, но обыкновенными линейными упругими ограничениями на внешнем краю. Краевые ограничения касаются ротационного и поперечнего движения. Особое внимание обращается на исследование эффекта поперечного ограничения на максиимумы реакции. В качестве практического примера, даются численные результаты, для пластинки подверженной действию И—образной волны давления в следствие звукового удара.